Final test answers (Wednesday 13 December 2023, 15:00-17:00 CET)

Elements of Mathematics – Bioinformatics for Health Sciences

1. Consider the following matrix:

$$M = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 0 & 4 & 4 \end{bmatrix}.$$

   (a) **(1 point)** Provide a basis of the column space of M. Justify your answer.

   Answer: One way to go about this is to conduct Gauss-Jordan elimination:

$$\begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 0 & 4 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \overset{a,b}{\sim} \begin{bmatrix} 1 & 0 & 0 \\ 3 & 2 & 2 \\ 0 & 4 & 4 \\ 1 & -2 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \overset{c}{\sim} \begin{bmatrix} 1 & 0 & 0 \\ 3 & 2 & 0 \\ 0 & 4 & 0 \\ 1 & -2 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix},$$

   using elementary column operations:

   a) $C_2 \leftarrow C_2 - 2C_1$

   b) $C_1 \leftarrow C_3 - 4C_1$

   c) $C_3 \leftarrow C_3 - C_2$

   Hence $\{(1,3,0),(0,2,4)\}$ is a basis of $C(M)$.

   (b) **(0.5 points)** Provide a basis of the null space of M. Justify your answer.

   Answer: After Gauss-Jordan elimination we know that the (yellow) vectors below the zero vector columns in the column echelon form matrix form a basis of $N(M)$. Hence $\{(-2,-1,1)\}$ is a basis of $N(M)$.

2. In this exercise we are going to find step-by-step the matrix of the symmetry of $\mathbb{R}^2$ with respect to the line L that goes through the origin forming an angle of $\pi/6$ radians with respect to the x-axis.

   (a) **(0.5 points)** Find a basis of $\mathbb{R}^2$ with respect to which the symmetry sought has the following matrix:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

   Answer: If $f : \mathbb{R}^2 \to \mathbb{R}^2$ denotes the symmetry sought, the basis $\{v_1, v_2\}$ must satisfy $f(v_1) = v_1$ and $f(v_2) = -v_2$. By definition of symmetry, vectors on the line L do not change and vectors perpendicular to L line get flipped. We can then take $v_1 = (\cos\pi/6, \sin\pi/6)$ and $v_2 = (-\sin\pi/6, \cos\pi/6)$.

   (b) **(0.5 points)** Compute the inverse of the decoding matrix B which has as columns the vectors of the basis you just found.

   Answer: Note that since the basis $\{v_1, v_2\}$ is orthonormal, the matrix B must be orthogonal, hence its inverse is simply its transpose. Therefore,

$$B^{-1} = B^t = \frac{1}{2}\begin{bmatrix} \sqrt{3} & 1 \\ -1 & \sqrt{3} \end{bmatrix}.$$

   (c) **(0.5 points)** Using all of the information above, compute the matrix of the symmetry with respect to L in coordinates of the canonical basis.

Answer: The matrix sought can be computed as:

$$S = BAB^{-1} = \frac{1}{4}\begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} \sqrt{3} & 1 \\ -1 & \sqrt{3} \end{bmatrix} = \frac{1}{4}\left(\begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix}\begin{bmatrix} \sqrt{3} & 1 \end{bmatrix} - \begin{bmatrix} -1 \\ \sqrt{3} \end{bmatrix}\begin{bmatrix} -1 & \sqrt{3} \end{bmatrix}\right)$$

$$= \frac{1}{4}\left(\begin{bmatrix} 3 & \sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix} - \begin{bmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & 3 \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$$

3. **(2 points)** Consider the set H of all possible solutions of the equation $x + y - z = 0$. H is a vector subspace of $\mathbb{R}^3$. Find an orthonormal basis of H.

   Answer: We can first find a basis of H via Gauss-Jordan elimination of the matrix $A = [1\ 1\ -1]$ since $H = N(A)$:

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \overset{a,b}{\sim} \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

   using the elementary column operations:

   a) $C_2 \leftarrow C_2 - C_1$

   b) $C_3 \leftarrow C_3 + C_1$

   The set $\{(-1,1,0),(1,0,1)\}$ is a basis of H, but it is not orthogonal. It might be convenient to re-scale them so that their length is one. Let $u = 1/\sqrt{2}(-1,1,0)$ and $v = 1/\sqrt{2}(1,0,1)$. We can get an orthogonal basis by applying the Gram-Schmidt method: if $\tilde{w} = v - (v \cdot u)u$ and $w = \tilde{w}/\|\tilde{w}\|$, then $\{u, w\}$ is an orthonormal basis of H. In numbers: $v \cdot u = -1/2$,

$$\tilde{w} = \frac{1}{\sqrt{2}}[(1,0,1) - (1/2,-1/2,0)] = \frac{1}{\sqrt{2}}[(1/2,1/2,1)]$$

$$\Rightarrow w = \frac{\tilde{w}}{\|\tilde{w}\|} = \frac{1}{\sqrt{6}}(1,1,2).$$

   A possible orthonormal basis of H is $\{1/\sqrt{2}(1,0,1), 1/\sqrt{6}(1,1,2)\}$.

4. **(1 point)** Find the Taylor approximation of order 2 of the function $f(x) = \sqrt{x^2+1}$ at $a = 0$.

   Answer: We must compute $f(a)$, $f'(a)$ and $f''(a)$.

$$f(0) = \sqrt{0^2+1} = 1$$

   Applying the chain rule:

$$f'(x) = \frac{1}{2\sqrt{x^2+1}}2x = \frac{x}{\sqrt{x^2+1}} = \frac{x}{f(x)} \Rightarrow f'(0) = 0$$

   Applying Leibniz rule:

$$f''(x) = \frac{1}{f(x)} - x\frac{f'(x)}{f(x)^2} \Rightarrow f''(0) = 1$$

   Therefore the Taylor approximation we sought is:

$$T(x) = f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 = 1 + \frac{1}{2}x^2.$$

5. If b is a positive real number, the *logarithm of x to base b*, denoted $\log_b(x)$, is a function that satisfies the following identity for all $x > 0$:

$$x = b^{\log_b(x)}.$$

The natural logarithm, which we simply denote $\log(x)$, is the logarithm to base e, where e is Euler's number.

For example, $\log_2(2^{100}) = 100$, $\log_5(25) = 2$ and $\log(e^3) = 3$.

(a) **(0.5 points)** Justify why the following identity holds for all $x > 0$?

$$b^x = e^{x \log(b)}$$

Answer:

$$b^x = (e^{\log(b)})^x = e^{x \log(b)}$$

(b) **(0.5 point)** Using the fact that $x = b^{\log_b(x)}$, compute the derivative of the function $\log_b(x)$?
Answer: Starting from the identity $x = b^{\log_b(x)}$, we can transform the second member using the hint of the previous question:

$$x = b^{\log_b(x)} = e^{\log(b) \log_b(x)}$$

Taking derivatives in both members of the equation:

$$1 = (e^{\log(b) \log_b(x)})',$$

but the second member can be transformed applying the chain rule:

$$(e^{\log(b) \log_b(x)})' = e^{\log(b) \log_b(x)}(\log(b) \log_b(x))' = e^{\log(b) \log_b(x)} \log(b)(\log_b(x))' = x \log(b)(\log_b(x))'$$

Therefore

$$(\log_b(x))' = \frac{1}{x \log(b)}$$

6. In this exercise we are going to study the critical points of the function

$$f(x, y, z) = x^2 + y^3 + z^2 - 2x - 4y - 6z.$$

(a) **(1 point)** Compute the critical points of f.
Answer:
Let's compute the first partial derivatives of f:

$$\frac{\partial f}{\partial x} = 2x - 2, \quad \frac{\partial f}{\partial y} = 3y^2 - 4 \quad \frac{\partial f}{\partial z} = 2z - 6$$

The critical points are those where both partial derivatives vanish, i.e., the points satisfying the three equations $2x = 2$, $3y^2 = 4$ and $2z = 6$.
There are two critical points $P_1 = (1, \frac{2}{\sqrt{3}}, 3)$ and $P_2 = (1, -\frac{2}{\sqrt{3}}, 3)$.

(b) **(1 point)** Compute the Hessian matrix of the only critical point of f that has all three coordinates positive (let's denote it P_1).
Answer:
Let's start by computing the second order partial derivatives of f:

$$\frac{\partial^2 f}{\partial x^2} = 2, \quad \frac{\partial^2 f}{\partial y^2} = 6y, \quad \frac{\partial^2 f}{\partial z^2} = 2$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial z} = \frac{\partial^2 f}{\partial x \partial z} = 0$$

The Hessian matrix of f is:

$$Hf(P_1) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2\sqrt{6} & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

(c) **(0.5 points)** What is the nature of the critical point P_1?

Answer: Since $Hf(P_1)$ is diagonal, its eigenvalues are just the elements of the diagonal, hence the eigenvalues of $Hf(P_1)$ are positive and P_1 is a local minimum.

(d) **(0.5 points)** What is the nature of the other critical point?

Answer: Since there are positive and negative eigenvalues of $Hf(P_2)$, P_2 is a saddle point.