Final Test (Monday 13th Dec 2021): Solutions
Elements of Mathematics – Bioinformatics for Health Sciences

1. **(2 points)** Suppose that A is a square matrix with $\det(A) \neq 0$.

   a) **(1 point)** Explain why $\mathrm{rank}(A^2) = \mathrm{rank}(A)$.

   b) **(1 point)** Find an example of a 2×2 matrix such that $\mathrm{rank}(A^2) < \mathrm{rank}(A)$.

   Answer:

   a) Both A and A^2 must be full rank square matrices since $\det(A) \neq 0$ and $\det(A^2) = \det(A)^2 \neq 0$.

   b) By virtue of the first point such an example must satisfy $\mathrm{rank}(A) = 1$, so we need to find a rank 1 matrix A such that $\mathrm{rank}(A^2) = 0$, in other words, A^2 is the zero matrix. Alternatively, one can search for a rank 1 matrix A such that $C(A) \subset N(A)$. Note that the matrices

   $$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

   would both work.

2. **(1.5 points)** Consider the following matrix:

   $$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

   a) **(0.5 point)** Find a basis of $C(A)$ the column space of A.

   b) **(0.5 point)** Find a basis of $N(A)$ the null space of A.

   c) **(0.5 point)** Find an orthonormal basis of $N(A)$.

   Answer:

   a) $\mathcal{C} = \{(1, 1, 1)\}$ this is immediate.

   b) Note that $\dim C(A) = 3 - \dim N(A)$ which enforces $\dim N(A) = 2$. The vectors $(1, 0, -1), (0, 1, -1)$ are easy to find members of $N(A)$ and are linearly independent, so they must also be a generating set, then $\mathcal{N} = \{(1, 0, -1), (0, 1, -1)\}$ is a basis of $N(A)$.

   c) Let $u = (1, 0, -1)$ and $v = (0, 1, -1)$. Note that $\mathcal{N} = \{u, v\}$ is not an orthonormal basis of $N(A)$ because $u \cdot v = 1 \neq 0$. One way to go is to substract from v its orthogonal projection onto $\mathrm{span}(u)$. Let's compute the matrix of this orthogonal projection. Taking $\tilde{u} = u/\mathrm{length}(u)$, we can build the sought orthogonal projection as:

   $$P = \tilde{u}\tilde{u}^t = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

   The non-zero vector $e = v - Pv$ is orthogonal to u:

   $$e = v - Pv = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} - \begin{bmatrix} 1/2 \\ 0 \\ -1/2 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1 \\ -1/2 \end{bmatrix}$$

   Therefore taking $\tilde{e} = e/\mathrm{length}(e) = (-1/\sqrt{6}, 2/\sqrt{6}, -1/\sqrt{6})$ we get $\mathcal{O} = \{\tilde{u}, \tilde{e}\}$ an orthonormal basis of $N(A)$.

3. **(1.5 points)** Consider a matrix $A \in \mathbb{R}^{n\times 3}$ with colum vectors $u, v, w \in \mathbb{R}^n$, i.e., $A = [\, u \,|\, v \,|\, w \,]$.

a) **(0.5 point)** Find a matrix $E \in \mathbb{R}^{3\times 3}$ such that $AE = [\, u \,|\, v \,|\, w - u - 2v \,]$.

b) **(0.5 point)** Find a matrix $P \in \mathbb{R}^{3\times 3}$ such that $AP = [\, w \,|\, u \,|\, v \,]$.

c) **(0.5 point)** Find P^{-1} the inverse matrix of P.

Answer:

a)

$$E = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

b)

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

c) P is an orthogonal matrix, so its inverse is its transpose. Alternatively, we may observe that P^{-1} should restore the original arrangement of columns so that $APP^{-1} = [\, u \,|\, v \,|\, w \,]$, which can be accomplished if $AP^{-1} = [\, v \,|\, w \,|\, u \,]$. In any case,

$$P^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

4. **(1.5 points)** Consider the following matrix:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}$$

a) **(0.5 point)** Find the eigenvalues of $\Omega = A^t A$.

b) **(0.5 point)** Find a basis of $\mathbb{R}^2$ of eigenvectors of Ω.

c) **(0.5 point)** Compute Ω^5.

Answer:

a)

$$\Omega = A^t A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

The eigenvalues of Ω must satisfy $\lambda_1 + \lambda_2 = \operatorname{tr}(\Omega) = 3 + 3 = 6$ and $\lambda_1\lambda_2 = \det(\Omega) = 9 - 1 = 8$, thus $\lambda_1 = 2$ and $\lambda_2 = 4$ work.

b)

$$\Omega - \lambda_1 I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

whence $N(\Omega - \lambda_1 I) = \operatorname{span}\{(1, -1)\}$

$$\Omega - \lambda_2 I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

whence $N(\Omega - \lambda_2 I) = \operatorname{span}\{(1, 1)\}$.
Then $\mathcal{B} = \{(1, -1), (1, 1)\}$ is a basis of $\mathbb{R}^2$ of eigenvectors of Ω.

c) We can factor Ω using the "decoder" matrix C, the "encoder" matrix C^{-1} and the diagonal matrix $\Lambda = \text{diag}(2,\, 4)$ as $\Omega = C\Lambda C^{-1}$. Then $\Omega^5 = C\Lambda^5 C^{-1}$.

$$C = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$C^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$\Omega^5 = C\Lambda^5 C^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 2^5 & 0 \\ 0 & 4^5 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2^5 & 4^5 \\ -2^5 & 4^5 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 4^5+2^5 & 4^5-2^5 \\ 4^5-2^5 & 4^5+2^5 \end{bmatrix}$$

5. **(1.5 point)** Consider the following function:

$$f(x, y) = \frac{1}{1 + e^{-ax-by}}$$

which results from composing the (single-variable) standard logistic function

$$\sigma(x) = \frac{1}{1 + e^{-x}}$$

and the two-variable linear function.

$$L(x, y) = ax + by,$$

where $a, b \in \mathbb{R}$ are parameters of the function.

a) **(0.5 point)** Verify that $\sigma'(x) = \sigma(x)(1 - \sigma(x))$

b) **(1 point)** Compute $\nabla f(0, 0)$, i.e., the gradient vector of f at the point (0,0).

Answer:

a) On the one hand, by using the chain rule we can compute the derivative of the logistic function:

$$\sigma'(x) = \frac{-1}{(1 + e^{-x})^2} \cdot (-e^{-x}) = \frac{e^{-x}}{(1 - e^{-x})^2}$$

On the other hand, we can develop the expression $\sigma(x)(1 - \sigma(x))$:

$$\sigma(x)(1 - \sigma(x)) = \frac{1}{1 + e^{-x}} \cdot \frac{e^{-x}}{1 + e^{-x}} = \frac{e^{-x}}{(1 - e^{-x})^2}$$

b)

$$\frac{\partial f}{\partial x} = \frac{-1}{(1 + e^{-ax-by})^2} \cdot (-ae^{-xa-by}) = \frac{ae^{-xa-by}}{(1 + e^{-ax-by})^2}$$

$$\frac{\partial f}{\partial y} = \frac{-1}{(1 + e^{-ax-by})^2} \cdot (-be^{-xa-by}) = \frac{be^{-xa-by}}{(1 + e^{-ax-by})^2}$$

Since the gradient is $\nabla f(x, y) = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$, $\nabla f(0, 0) = (a/4, b/4)$.

6. **(2 points)** Determine the nature of the critical point $(0, 0)$ of the fuction

$$f(x, y) = x^3 + 2xy^2 + x^2 + 3xy + y^2 + 1.$$

Answer:

Let's compute the partial derivatives, then the eigenvalues of the Hessian matrix of f at $(0, 0)$:

$$\frac{\partial f}{\partial x} = 3x^2 + 2y^2 + 2x + 3y \qquad \frac{\partial f}{\partial y} = 4xy + 3x + 2y$$

$$\frac{\partial^2 f}{\partial x^2} = 6x + 2 \qquad \frac{\partial^2 f}{\partial x \partial y} = 4y + 3 \qquad \frac{\partial^2 f}{\partial y^2} = 4x + 2$$

Then the Hessian of f at $(0, 0)$ is the matrix:

$$H = Hf(0, 0) = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}.$$

The eigenvalues of H must satisfy $\lambda_1 + \lambda_2 = \text{tr}(H) = 2 + 2 = 4$ and $\lambda_1 \lambda_2 = \det(H) = 4 - 9 = -5$, thus $\lambda_1 = -1$ and $\lambda_2 = 5$ work. Since $\lambda_1 \lambda_2 < 0$, $(0, 0)$ is neither a local maximum nor a local minimum, but a saddle point.