4

Linear encoding and decoding

> Without music, life would be a mistake
>
> F. Nietzsche

We have already seen that once we are working with a basis, all vectors can be represented with the coordinates in this basis. We also know that a vector space admits many different bases, which, as we will see gives us a lot of room for representing vectors in ways that are more easily interpretable and self-correcting.

However, we are now left with the problem of how to switch from one representation to another. In this chapter we will be dealing with change of coordinates. We will see that both vectors and linear maps can be subject to change of coordinates.

4.1 Change of basis

Suppose that we are given two different bases of V, say $\mathscr{A} = \{a_1, \ldots, a_n\}$ and $\mathscr{B} = \{b_1, \ldots, b_n\}$. Suppose that the coordinates of v in the base $\mathscr{A}$ are $v_{\mathscr{A}} = (\alpha_1, \ldots, \alpha_n)$. What are the coordinates of v in the base $\mathscr{B}$?

4.1.1 Decode

To reason step by step, let's consider first the situation where we are given the vector v expressed in coordinates of $\mathscr{A}$, say $v_{\mathscr{A}} = (\alpha_1, \ldots, \alpha_n)$. We want to express v in our default vector representation, namely, using coordinates in the canonical basis.

By definition of coordinates, we know that

$$v = \alpha_1 a_1 + \ldots + \alpha_n a_n$$

This means that if we build a matrix A by putting the vectors of $\mathscr{A}$ – expressed in the canonical basis – as columns, i.e. $A = [a_1 \mid \ldots \mid a_n]$ and multiply it by $v_{\mathscr{A}} = (\alpha_1, \ldots, \alpha_n)$, the resulting vector will contain the coordinates of v in the canonical basis, because:

$$v = \alpha_1 a_1 + \ldots + \alpha_n a_n = A v_{\mathscr{A}}$$

We have just defined a function that converts $v_{\mathscr{A}}$ into v: it is just multiplying by A!

For convenience we will refer to it as a *decoder*, because it brings us back our default representation using coordinates in the canonical basis.

4.1.2 Encode

Now let's consider the situation where we are given the vector v expressed in coordinates of the canonical basis – our default – and we want to express it in coordinates of the basis $\mathscr{B}$. Using the same reasoning as before, if we build a matrix B by putting the vectors of $\mathscr{B}$ – expressed in the canonical basis – as columns and multiply it by the coordinates we want to compute $v_{\mathscr{B}} = (\beta_1, \ldots, \beta_n)$, the following identity must hold

$$v = Bv_{\mathscr{B}}.$$

Since v is known and $v_{\mathscr{B}}$ is unknown, we are bound to solve the system of linear equations. But this is equivalent to multiplying by the inverse of B – we know that B is invertible because it is full-rank:

$$v_{\mathscr{B}} = B^{-1}v.$$

We have just defined the function that converts v into $v_{\mathscr{B}}$: it is just multiplying by B^{-1}! For convenience we will refer to it as the *encoder*, because it replaces our default representation by a custom one.

4.1.3 Decode then Encode

Back to our original problem, we want to convert $v_{\mathscr{A}}$ into $v_{\mathscr{B}}$. What to do? We can break down our calculation in two steps:

- Decode $v_{\mathscr{A}}$ into the canonical basis: $v = Av_{\mathscr{A}}$
- Encode v into the basis $\mathscr{B}$: $v_{\mathscr{B}} = B^{-1}v$

Putting the pieces together, we have come up with the change of basis function: it is multiplying by the matrix $B^{-1}A$:

$$v_{\mathscr{B}} = B^{-1}Av_{\mathscr{A}}.$$

4.2 Linear maps seen through different bases

Coordinate changes also modify the way we represent linear maps with matrices.

4.2.1 Lifting a linear map

Consider a linear map associated with the matrix M using the default vector representation – coordinates in the canonical basis:

$$\begin{aligned} F\colon V &\longrightarrow W \\ v &\longmapsto Mv \end{aligned}$$

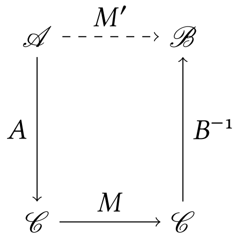
What would the matrix of F be if now we want the input and output be given in coordinates of $\mathscr{A}$ and $\mathscr{B}$, respectively?

Using our encoding-decoding trick, we can proceed in three steps:

- Decode $v_{\mathscr{A}}$ into the canonical basis: $v = Av_{\mathscr{A}}$
- Now we can do $F(v) = Mv$
- Encode Mv into the basis $\mathscr{B}$ by doing $B^{-1}Mv$

All in all, the matrix of F in the new reference bases is $M' = B^{-1}MA$.

We can visually aprehend what just happen in the following diagram:



4.2.2 Dipping a linear map

Now consider a linear map associated with the matrix M' that is already expressed in coordinates of bases $\mathscr{A}$ and $\mathscr{B}$ different from the canonical:

$$\begin{aligned} F\colon V &\longrightarrow W \\ v_{\mathscr{A}} &\longmapsto (M' v_{\mathscr{A}})_{\mathscr{B}} \end{aligned}$$

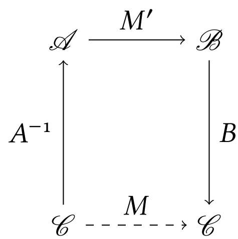
What would the matrix of F be if now we want the input and output be given in coordinates of the canonical basis?

Using our encoding-decoding trick, we can proceed in three steps:

- Encode v into the basis $\mathscr{A}$: $v_{\mathscr{A}} = A^{-1}v$
- Now we can do $F(v_{\mathscr{A}}) = M'v_{\mathscr{A}}$
- Decode $M'v_{\mathscr{A}}$ from the basis $\mathscr{B}$ back into the canonical basis by doing $BM'v_{\mathscr{A}}$

All in all, the matrix of F in the canonical basis is $M = BM'A^{-1}$.

We can visually aprehend what just happen in the following diagram:



4.3 Crafting linear maps

One of the conclusions we were drawn to in the previous sections is that linear maps identify with matrices as long as a basis is specified for the input and output vector spaces that the linear transformation connects.

At some point it will be very useful to construct the matrix of a linear transformation just by knowing the effect it has in some vectors. The good news is that if we know the effect that a given linear transformation has on the vectors of a basis of the input vector space, then a linear transformation with this property exists and it is uniquely specified.

Proposition 15. *If $\{u_1, \ldots, u_n\}$ is a basis of V and $w_1, \ldots, w_n$ are some vectors in another vector space W, then there is a linear transformation $f : V \to W$ such that $f(v_i) = w_i$ for all*

$1 \leq i \leq n$.

To prove it we will just describe what is the matrix of the linear transformation in some appropriate coordinates. If the coordinates of the $w_1, \ldots, w_n$ in some basis $\mathscr{B}$ of W are known, we can define f as the linear transformation whose matrix, using coordinates of $\mathscr{A}$ in V and $\mathscr{B}$ in W, whose columns are the w_j's.

4.4 TL;DR

Vectors and linear transformations can be see through the lens of different bases. Whenever we change a basis, the coordinates of the vectors are bound to change – this is commonly referred to as doing a change of coordinates. Because of these differences, linear transformations can also be represented by many different matrices. By applying encoding and decoding transformations in the input and output vector spaces of linear transformations, we can accomplish these changes of coordinates in a systematic and easy-to-remember way. Looking at the same linear transformation through different coordinate systems can be quite handy, because it can lead to matrix representations that are much easier to read and understand.