EXERCISES: SESSION 3

1. Elementary column operations can be accomplished by matrix multiplication with suitable matrices. Suppose we have some 4×3 input matrix A. Find out which elementary column operations are accomplished after doing the matrix multiplication AE, for the following matrices E:

(1)

$$E = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(2)

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

(3)

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

2. Compute the inverse of the matrix

$$A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$$

using Gauss-Jordan elimination.

3. Let $A \in \mathbb{R}^{n \times n}$ be a square matrix. We say that R is a right-inverse of A if $AR = \mathrm{Id}_{n \times n}$. Analogously, we say that L is a left-inverse matrix of A if $LA = \mathrm{Id}_{n \times n}$.

(1) What are the sizes of R and L, respectively?
(2) Show that, whenever they exist, the left and right inverses are unique.
(3) Show that if R and L are right and left inverses of $A \in \mathbb{R}^{n \times n}$, respectively, then $R = L$.

4. Compute the inverse of the matrix

$$A = \frac{1}{2} \begin{bmatrix} 1 & 0 & -\sqrt{3} \\ 0 & 1 & 0 \\ \sqrt{3} & 0 & 1 \end{bmatrix}$$

using Gauss-Jordan elimination.

5. Let

$$A = \begin{bmatrix} -4 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Let $f_A : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map defined as $f_A(v) = Av$.

(1) What is the rank of the matrix A? Justify your answer.
(2) Let $\{e_1 = (1,0,0), e_2 = (0,1,0), e_3 = (0,0,1)\}$ be the canonical basis of $\mathbb{R}^3$. Compute $f(e_1)$, $f(e_2)$ and $f(e_3)$.
(3) Give a basis of the vector subspace $S \subset \mathbb{R}^3$ generated by $f(e_1), f(e_2), f(e_3)$.

6. Let

$$M = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \end{bmatrix}.$$

(1) How can you know whether M is invertible? Justify your answer.
(2) Compute M^{-1} using Gauss-Jordan elimination.
(3) Does the linear map $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by M preserve orientation?
(4) How does the linear map f transform areas?

7. Determinants are compatible with matrix multiplication, in the sense that if $A, B \in \mathbb{R}^{n\times n}$ are square matrices, then $\det(AB) = \det(A)\det(B)$. Consider the matrices:

$$A = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 2 & 2 \\ 0 & 0 & 7 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & -2 \end{bmatrix}$$

(1) Using the recursive definition of determinant, reason why the determinant of a diagonal matrix is the product of the entries in the diagonal, i.e. if $D = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$ then $\det(D) = \lambda_1 \cdot \dots \cdot \lambda_n$.
(2) Compute the determinant of the identity matrix.
(3) Notice that the matrices A and B given above satisfy that all their entries below the diagonal are zero: matrices satisfying this condition are known as "upper triangular" matrices. Reason why the determinant of an upper triangular matrix is the product of the entries in the diagonal, i.e. if $T = (a_{ij})$ is upper triangular, then $\det(T) = a_{11} \cdot \dots \cdot a_{nn}$.
(4) Verify that the opening remark holds for the matrices A and B given above, that is, $\det(AB) = \det(A)\det(B)$.
(5) Recall that when A is an invertible matrix, it satisfies $AA^{-1} = A^{-1}A = I$. Making use of the opening remark if necessary, reason why $\det(A^{-1}) = 1/\det(A)$.

8. Let

$$A = \begin{bmatrix} -4 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Let $f_A : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map defined as $f_A(v) = Av$.

(1) What is the rank of the matrix A? Justify your answer.
(2) Let $\{e_1 = (1,0,0), e_2 = (0,1,0), e_3 = (0,0,1)\}$ be the canonical basis of $\mathbb{R}^3$. Compute $f(e_1)$, $f(e_2)$ and $f(e_3)$.
(3) Give a basis of the vector subspace $S \subset \mathbb{R}^3$ generated by $f(e_1), f(e_2), f(e_3)$.

9. A linear map $f : \mathbb{R}^n \to \mathbb{R}^n$ is referred to as a projection if it verifies $f \circ f = f$, i.e., $f(f(v)) = f(v)$ for any vector v.

(1) Prove that if f is a projection, then $\mathrm{Id} - f$ is also a projection.
(2) If f is a projection, is $f - \mathrm{Id}$ a projection?
(3) Is the identity map Id a projection?
(4) Is the zero map Z a projection?
(5) Is the linear map defined by the matrix
$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
a projection?
(6) Is the linear map defined by the matrix
$$A = \frac{1}{2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$
a projection?
(7) Is the linear map defined by the matrix
$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$
a projection?

10. In general, you can craft your own projection $f : \mathbb{R}^n \to \mathbb{R}^n$ in three easy steps:

(1) Pick any vector subspace $S \subset \mathbb{R}^n$.
(2) Choose a basis of S, namely $s_1, \ldots, s_k$, and extend it to a basis $\mathcal{B}$ of $\mathbb{R}^n$.
(3) Define a linear map that maps the basis vectors of S to 0 and the other basis vectors of $\mathcal{B}$ to themselves.