EXERCISES: SESSIONS 13-14

1. Compute the local quadratic approximation of the following functions at the indicated points:

(1) $f(x) = 1 - (x^2 + y^2)$ at $a = (0, 0)$ and at $a = (1, 2)$.
(2) What is the relationship between the two quadratic approximations computed in the previous step?
(3) $f(x) = x^3 - x + y^3 - y$ at $a = (\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3})$.

2. Consider a general quadratic polynomial in two variables:

$$p(x, y) = a + a_x x + a_y y + a_{xx} x^2 + a_{xy} xy + a_{yy} y^2$$

where $a, a_x, a_y, a_{xx}, a_{xy}, a_{yy} \in \mathbb{R}$.

(1) Compute all the first partial derivatives of $p(x, y)$.
(2) Compute all the second partial derivatives of $p(x, y)$. Do I need to specify a point to evaluate these derivatives?
(3) Compute the local quadratic approximations at $a = (0, 0)$ and at $a = (1, 2)$.
(4) What happens if you develop both expressions until you reach the general polynomial form? Do they look similar? Can you elaborate an explanation of what is going on here?

3. Study the critical points of the functions of Exercise 1. For each function, follow the steps:

(1) Determine all the critical points.
(2) Compute the Hessian matrix at each critical point.
(3) Compute the eigenvectors and eigenvalues of the Hessian matrices.
(4) Finally, apply the quadratic classification criteria to determine maxima, minima and saddle points if possible.

4. Consider the following function $\phi : \mathbb{R}^2 \to \mathbb{R}$:

$$\phi(\mathbf{x}) = \phi(x, y) = \frac{1}{2\pi} (\det \Omega)^{-\frac{1}{2}} \mathrm{e}^{-\frac{1}{2}(\mathbf{x}-\mu)^t \Omega (\mathbf{x}-\mu)}$$

where μ and Ω are:

$$\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} \text{ and } \Omega = \begin{bmatrix} \omega_x & \omega_{xy} \\ \omega_{xy} & \omega_y \end{bmatrix}.$$

This function ϕ is in fact the probability density function of the bivariate Gaussian distribution.

(1) Compute the partial derivatives and Jacobian matrix of ϕ at the points $(0, 0)$ and μ, respectively.
(2) Compute the Hessian matrix of ϕ at μ, say $H\phi(\mu)$.
(3) Compute the eigenvectors and eigenvalues of $H\phi(\mu)$.

Whenever possible, try to provide your solution in an as compact expression as possible.

5. Study the critical points of the "residual sum of squares" function $\psi : \mathbb{R}^n \to \mathbb{R}$ defined as

$$\psi(\mathbf{x}) = (A\mathbf{x} - b)^t (A\mathbf{x} - b),$$

where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$.

(1) What are the partial derivatives of of the function $f(\mathbf{x}) = (w^t\mathbf{x} - c)$, with $w \in \mathbb{R}^n$ and $c \in \mathbb{R}$.

(2) What are the partial derivatives of of the function $f(\mathbf{x}) = (w^t\mathbf{x} - c)^2$, with w, c are as above.

(3) Prove that the expression defining $\psi(\mathbf{x})$ can be written as:

$$(w_1^t\mathbf{x} - b_1)^2 + (w_2^t\mathbf{x} - b_2)^2 + \ldots + (w_m^t\mathbf{x} - b_m)^2$$

where $w_1, \ldots, w_m \in \mathbb{R}^n$ are the row vectors of A.

(4) In view of the previous remark, can you compute the partial derivatives of $\psi(\mathbf{x})$?

(5) Prove that the jacobian matrix is given by:

$$J\psi(\mathbf{x}) = 2(A\mathbf{x} - b)^t A.$$

(6) Compute the critical point(s) of ψ.

(7) Do the second partial derivatives of ψ (hence the Hessian matrix) differ between points? Why?

(8) Compute the Hessian matrix of ψ, say $H\psi$. Verify that $H\psi = 2A^tA$.

(9) Observe that if A is full-rank, the Hessian of ψ must be symmetric, positive-definite, meaning that all its eigenvalues $\lambda_1, \ldots, \lambda_n > 0$. What does this tell you about the classification of the critical point(s) of ψ?

(10) Provide an expression for the local quadratic approximation of ψ at the critical point(s). Try to come up with a compact, matrix-algebra expression.