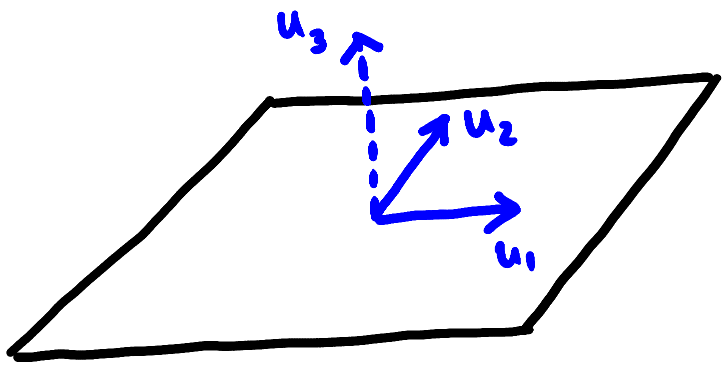
Exercise 6 (session 5)

$f : \mathbb{R}^3 \to \mathbb{R}^3$ reflection w.r.t. plane

$S = \text{span}\{(1,1,0), (1,0,-1)\}$



$S \subset \mathbb{R}^3$

$\mathcal{B} = \{u_1, u_2, u_3\}$ basis of $\mathbb{R}^3$ where

u_1, u_2 span S

u_3 perpendicular to S

In coordinates of the basis $\mathcal{B}$

the matrix of f is
$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

We can take $u_1 = (1,1,0)$

$u_2 = (1,0,-1)$

u_3 must be chosen perpendicular to u_1 and u_2

$$u_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ such that } \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Leftrightarrow u_3 = \lambda \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

We choose $u_3 = (1,-1,1)$

Compute the matrix of f in the canonical basis

$$\begin{tikzcd} \mathcal{B} \arrow[r, "R"] & \mathcal{B} \arrow[d, "C"] \\ \text{Canonical} \arrow[u, "C^{-1}"] \arrow[r, "A"'] & \text{Canonical} \end{tikzcd}$$

where $C = \left[u_1 | u_2 | u_3 \right] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{bmatrix}$

$A = CRC^{-1}$

Compute C^{-1} (shiny app ?)

$$C^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & -2 \\ 1 & -1 & 1 \end{bmatrix}$$

$$A = \frac{1}{3}\underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}}\begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & -2 \\ 1 & -1 & 1 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \\ 0 & -1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & -2 \\ 1 & -1 & 1 \end{bmatrix} = \boxed{\frac{1}{3}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ -2 & 2 & 1 \end{bmatrix}}$$

matrix of the reflection w.r.t. S in coords. of the canonical basis

Final check:

$$Au_1 = \frac{1}{3}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ -2 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = u_1 \checkmark$$

$$Au_2 = \frac{1}{3}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ -2 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 3 \\ 0 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = u_2 \checkmark$$

$$Au_3 = \frac{1}{3}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ -2 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -3 \\ 3 \\ -3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} = -u_3 \checkmark$$